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Constrained systems and analytical mechanics in spaces with torsion

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Abstract. A system with anholonomic constraints where the trajectories of physical degrees of freedom are autoparallels on a manifold equipped with a general Cartan connection is discussed. A variational principle for the autoparallel trajectories is derived from the d'Alambert–Lagrange principle for anholonomic constrained systems. A geometrical (coordinate-independent) formulation of the variational principle is given. Its relation to Sedov's anholonomic variational principle for dissipative systems and to Poincaré's variational principle in anholonomic reference frames is established. A modification of Noether's theorem due to the torsion force is studied. A non-local action whose extrema contain the autoparallels is proposed. The action can be made local by adding auxiliary degrees of freedom coupled to the original variables in a special way.

1. Anholonomic constrained systems

There is no need to explain the importance of constrained systems in modern physics (e.g. electrodynamics, Yang–Mills theory, general relativity, etc). Constraints in dynamical systems are usually regarded as a part of the Euler–Lagrange equations of motion which do not involve time derivatives of order higher than 1. In other words, both constraints and equations of motion result from the least action principle applied to some Lagrangian. The existence of the Lagrangian formalism is of great importance in constrained systems because it allows one to develop the corresponding Hamiltonian formalism [1] and canonically quantize the system [1]. Yet, the variational principle is a powerful technical tool to find integrals of motion of dynamical systems via symmetries of the Lagrangian.

The Hamiltonian (or Lagrangian) constrained systems form a relatively small class of constrained dynamical systems. Given an 'unconstrained' system whose dynamics is governed by a Lagrangian $L = L(v, x)$, v^i and x^i being generalized velocities and coordinates, respectively, one can turn it into a constrained system by imposing supplementary conditions $F_\alpha(v, x) = 0$ (constraints) which has to be fulfilled by the actual motion of the system. There two ways to incorporate the constraints into a dynamical description. First, one can simply modify the Lagrangian $L \to L + \lambda^\alpha F_\alpha$ with λ^α being the Lagrange multipliers and treat the latter as independent dynamical variables. In doing so, we are led to the Lagrangian constrained dynamics. The other way is to supplement the unconstrained Euler–Lagrange equations $\mathrm{d}/\mathrm{d}t\,(\partial_v L) - \partial_x L = 0$ by the constraints $F_\alpha = 0$. It is well known that if the constraints are not integrable, the two dynamical descriptions are not equivalent [2, 3]. The non-integrable constraints are called anholonomic, and the dynamical

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systems described in the latter of the above two ways are known as anholonomic systems. In general, there exists no Lagrangian or Hamiltonian formalism for anholonomic systems, i.e. they are *non-Lagrangian* dynamical systems [2]. The existence of constraints implies that the dynamical system has non-physical degrees of freedom, meaning that the actual motion of the system is determined by a lesser number of independent parameters than the number of initial values of generalized coordinates and velocities. For holonomic systems, the motion of the physical degrees of freedom can be obtained by applying the conventional variational principle to the Lagrangian reduced on the constraint surface $L \to L|_{F=0}$. When applied to anholonomic constraints, this procedure leads to wrong equations of motion [2, 3]. Despite of being non-Lagrangian, the dynamics of physical degrees of freedom in anholonomic constrained systems may possess 'good' properties, e.g. a covariance under some group transformations, existence of integrals of motion with a clear physical interpretation, etc, that is, the properties that one always wants to see in physical systems.

An example of this kind is provided by the autoparallel and geodesic motions on a manifold equipped with a general connection $\Gamma^{\mu}{}_{\nu\sigma}$ compatible with metric $g_{\mu\nu}$, $D_\mu g_{\nu\sigma} = 0$, where D_μ is the covariant derivative. In [4] it has been shown that the autoparallels can be realized as the trajectories of the physical degrees of freedom in a special anholonomic constrained system, while the geodesics can always be regarded as the trajectories in a holonomic constrained system. On a manifold M the autoparallels and geodesics are determined respectively by the following equations covariant under general coordinate transformations

$$D_v v^\mu = \dot{v}^\mu + \Gamma^{\mu}{}_{\nu\sigma} v^\nu v^\sigma = 0 \tag{1.1}$$

$$\bar{D}_v v^\mu = \dot{v}^\mu + \bar{\Gamma}^{\mu}{}_{\nu\sigma} v^\nu v^\sigma = 0. \tag{1.2}$$

Here D_v is the covariant derivative along the velocity vector v^μ and $\bar{\Gamma}^{\mu}{}_{\nu\sigma}$ are the Christoffel symbols. Both equations (1.1) and (1.2) determine a curve that parallel-transports its tangent vector along itself. The curve with such a property is the autoparallel when the most general connection compatible with the metric is used to specify the parallel transport. The geodesic motion occurs if the natural Riemannian connection (induced by the metric) is chosen to define the parallel transport. The difference of (1.1) from (1.2) resides in the torsion force term. Any connection compatible with the metric can always be represented in the form [5] $g_{\sigma\lambda}\Gamma^{\lambda}{}_{\mu\nu} = \bar{\Gamma}_{\sigma\mu\nu} + K_{\sigma\mu\nu}$, where $K_{\sigma\mu\nu} = S_{\sigma\mu\nu} - S_{\mu\nu\sigma} + S_{\nu\sigma\mu}$ is called the contorsion tensor, and $g^{\sigma\lambda}S_{\lambda\mu\nu} = S^{\sigma}{}_{\mu\nu} = \frac{1}{2}(\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\nu\mu})$ is the torsion tensor. The deviation of the autoparallels from the geodesics is caused by the torsion force $K_{\mu\nu\sigma}v^\nu v^\sigma$.

The geodesic equation (1.2) follows from the Hamilton variational principle applied to the action

$$S = \tfrac{1}{2}\int \mathrm{d}t\, g_{\mu\nu}v^\mu v^\nu \tag{1.3}$$

or to its relativistic analogue

$$S = -\int \mathrm{d}s = -\int \mathrm{d}t\, \sqrt{g_{\mu\nu}v^\mu v^\nu} \tag{1.4}$$

with s being the proper time (or length) of the trajectory (in this case, $v^\mu = \mathrm{d}q^\mu/\mathrm{d}s$ and $\dot{v}^\mu = \mathrm{d}v^\mu/\mathrm{d}s$ in (1.2)). A particle moving along the autoparallel trajectory is an example of a non-Lagrangian system. A system of differential equations of second order is called Lagrangian if there exists a local Lagrangian such that the original system is equivalent to the Euler–Lagrange equations. One can formulate the sufficient conditions, known as the Helmholtz conditions, for a given dynamical system to be Lagrangian [6]. Even if these conditions are not fulfilled, one can still try to find a *non-local* action for a given dynamical

system. In section 6 we construct an explicit example of a non-local action whose extrema contain the autoparallels. From the geometrical point of view equation (1.1) is just as good as equation (1.2) and may be regarded or postulated as an equation of motion of a spinless particle on a manifold†.

Here we generalize the embedding procedure of [4] to arbitrary spaces with curvature and torsion. Next we use the representation of the autoparallel motion as a motion of an anholonomic system to establish a variational principle for the Lagrangians (1.3) and (1.4) which leads to equation (1.1). The variational principle is derived from the well known variational principles for anholonomic systems, such as Gauss' principle of least constraint, Hölder and d'Alambert–Lagrange principles [2, 3, 10]. However, in contrast to them, it has an advantage in that it applies to Lagrangians *reduced* on the surface of constraints. We shall also show that the new variational principle can be given a completely covariant (coordinate-independent) formulation on a manifold with a general Cartan connection. For this reason we shall refer to it as a *covariant* variational principle. Its relation to the variational principle of Poincaré [11] and to that proposed by Sedov for dissipative systems [12] is explained. Finally, we propose a modification of the actions (1.3) and (1.4) by adding new auxiliary degrees of freedom so that the modified actions have extrema being the autoparallels and admit the conventional Hamiltonian formalism.

2. Autoparallels from constrained motion

Consider a metric manifold M and local coordinates q^μ on it. Let $\Gamma^\mu{}_{\nu\sigma}$ be components of a connection on M in the coordinate basis. We denote $\mathcal{P}(M)$ the space of all paths in M, T_qM the tangent space at a point q^μ, and TM the tangent bundle. Consider an auxiliary Euclidean space $\mathbb{R}^n$ of the dimension greater than that of M, $n > \dim M$. Cartesian coordinates in $\mathbb{R}^n$ are denoted by x^i. In the space $\mathcal{P}(\mathbb{R}^n)$ of all paths in $\mathbb{R}^n$ we define a subspace of *conceivable* paths (i.e. of those allowed by constraints) as an image of $\mathcal{P}(M)$ in the embedding $\mathcal{P}(M) \to \mathcal{P}(\mathbb{R}^n)$:

$$x^i(s) = \int^s \mathrm{d}q^\mu \, \varepsilon^i_\mu(q) \tag{2.1}$$

for any path $q^\mu(s)$ in M. The embedding functions $\varepsilon^i_\mu(q)$ are smooth on M. From the path embedding (2.1) follows the embedding of the tangent space T_qM into $\mathbb{R}^n$

$$v^i = \varepsilon^i_\mu(q)v^\mu \qquad v^\mu \in T_qM \tag{2.2}$$

because one can always find a curve $q^\mu(s)$ passing through a point q^μ such that $v^\mu = \mathrm{d}q^\mu(s)/\mathrm{d}s$.

The space M cannot be embedded into $\mathbb{R}^n$ pointwise if the constraints (2.2) on the tangent space (or on the velocities of the conceivable motion) are not integrable (anholonomic constraints):

$$\oint \mathrm{d}q^\mu \, \varepsilon^i_\mu(q) \neq 0 \tag{2.3}$$

for any closed path in M. That is, the one-forms $\varepsilon^i_\mu \mathrm{d}q^\mu$ are not closed, and there exist no mapping $M \to \mathbb{R}^n$, $x^i = x^i(q)$, induced by (2.2).

† Here we do not discuss physical arguments which of those two trajectories should be identified with a physical trajectory of a spinless point-like particle moving in a space with torsion. An interested reader may find physical arguments supporting both geodesics [7] and autoparallels [8, 9]. Note also that a massive particle with spin would follow neither of these trajectories due to the interaction between its spin and the spacetime geometry.

The scalar product in $\mathbb{R}^n$, $(w, v) = \delta_{ij} w^i v^j$, induces the metric on M. For any two vectors being the images of two elements of $T_q M$ we have

$$(w, v) = g_{\mu\nu} w^\mu v^\nu \qquad g_{\mu\nu} = (\varepsilon_\mu, \varepsilon_\nu) \qquad v^\mu, w^\mu \in T_q M. \tag{2.4}$$

Next, we determine a connection induced by the path embedding. To do so, we require that the parallel transport in M must be compatible with the embedding (2.1). This means the following. Consider a curve $q^\mu(s)$ passing through a point $q^\mu \in M$ and its image $x^i(s)$. Take a vector field $w^\mu \in T_q M$ and parallel-transport it along the curve $q^\mu(s)$. The resulting vector is then embedded into $\mathbb{R}^n$ by means of (2.2). Let us repeat the procedure in the opposite way. First we embed the vector w^μ and then parallel-transport its image along $x^i(s)$, the image of $q^\mu(s)$. The compatibility condition implies that the vectors obtained by these procedures may differ by a vector orthogonal to the hyperplane being the image of $T_q M$, and this should hold for any curve and any w^μ. We remark that this condition is weaker than a similar condition of [4] and it is sufficient to obtain the most general spaces with curvature and torsion.

An infinitesimal variation of w^μ under the parallel transport is proportional to $D_v w^\mu$, where $v^\mu = \dot{q}^\mu$, while the corresponding variation of its image, $w^i = \varepsilon^i_\mu w^\mu$, is $D_v w^i = \mathrm{d}_v w^i = \mathrm{d}w^i/\mathrm{d}s$ because the connection in $\mathbb{R}^n$ is trivial. The compatibility condition means that

$$\frac{\mathrm{d}w^i}{\mathrm{d}s} - \varepsilon^i_\mu D_v w^\mu = f^i_{\mu\nu} v^\mu w^\nu \qquad (\varepsilon_\sigma, f_{\mu\nu}) = 0. \tag{2.5}$$

Computing the derivative in the left-hand side of (2.5) and assuming v^μ and w^μ to be arbitrary we find

$$D_\mu \varepsilon^i_\nu = f^i_{\mu\nu}. \tag{2.6}$$

Multiplying this equation by ε^i_σ we obtain the connection coefficients

$$\Gamma^\mu{}_{\nu\sigma} = g^{\mu\lambda}(\varepsilon_\lambda, \partial_\nu \varepsilon_\sigma) \tag{2.7}$$

where $g^{\nu\mu} g_{\mu\sigma} = \delta^\nu_\sigma$. It is easy to verify that the compatibility condition also holds for tensors of higher rank, provided the induced metric is used to lower and rise tensor indices. Thanks to the compatibility condition, the induced connection turns out to be compatible with the induced metric $D_\mu g_{\nu\sigma} = (D_\mu \varepsilon_\nu, \varepsilon_\sigma) + (\varepsilon_\nu, D_\mu \varepsilon_\sigma) = 0$. In the non-integrable case (2.3), the induced connection may have a torsion

$$S^\mu{}_{\nu\sigma} = \tfrac{1}{2} g^{\mu\lambda} [(\varepsilon_\lambda, \partial_\nu \varepsilon_\sigma) - (\varepsilon_\lambda, \partial_\sigma \varepsilon_\nu)]. \tag{2.8}$$

The tensor fields $f^i_{\mu\nu}$ must also satisfy the integrability condition for equation (2.6) resulting from the commutation relation

$$[D_\mu, D_\nu] = R_{\mu\nu}{}^{\sigma\lambda} L_{\sigma\lambda} - 2 S^\sigma{}_{\mu\nu} D_\sigma \tag{2.9}$$

where $L_{\mu\nu} = -L_{\nu\mu}$ are generators of a (pseudo)orthogonal group acting in the tangent space of M, and $R_{\mu\nu}{}^{\sigma\lambda}$ is the Riemann–Cartan curvature tensor. Using the integrability conditions we derive

$$(f_{\mu\lambda}, f_{\nu\sigma}) - (f_{\nu\lambda}, f_{\mu\sigma}) = 2 R_{\mu\nu\lambda\sigma}. \tag{2.10}$$

That is, the tensor fields $f^i_{\mu\nu}$ determine the Riemann–Cartan curvature tensor on M. The case $f^i_{\mu\nu} = 0$ corresponds to the so-called teleparallel spaces with zero curvature and non-zero torsion. With sufficiently large n (at least one should have $2n > 1 + (\dim M)^2$, the number of independent components of the metric and torsion tensors), one can construct the most general connection on M.

Let us analyse dynamics induced by the anholonomic constraints (2.2), assuming the unconstrained motion to be a free motion in the Euclidean space $\mathbb{R}^n$. There are several variational principles for anholonomic systems [10] that can be applied to our system to derive the equations of motion. If the constraints are integrable, they are all equivalent to the conventional Hamilton variational principle.

In the auxiliary Euclidean space the states of the system are labelled by pairs $\psi = (v^i, x^i)$. For any two trajectories $x^i_{1,2}(s)$ passing through the state ψ, Gauss' deviation function is defined as [3, 2]

$$G_\psi = \tfrac{1}{2}(\dot v^i_1 - \dot v^i_2) H_{ij} (\dot v^j_1 - \dot v^j_2) \tag{2.11}$$

where $H_{ij}(\psi) = \partial^2 L / \partial v^i \partial v^j$ is the Hessian for a Lagrangian L in the state ψ. Gauss' principle of least constraint states that the deviation of conceivable motions (allowed by the constraints) from the released (unconstrained) motion takes a stationary value on the actual motion. A physical meaning of Gauss' principle is transparent: the acceleration (or the force) caused by the constraints must have a minimal deviation from the acceleration of the unconstrained motion. In our system the released motion is the free motion $\dot v^i = 0, H_{ij} = \delta_{ij}$, the accelerations of the conceivable motion are obtained by taking a time derivative of (2.2). The Gauss deviation function assumes the form

$$G_\psi = \tfrac{1}{2} g_{\mu\nu} D_v v^\nu D_v v^\mu + \tfrac{1}{2}(f_v, f_v) \qquad f^i_v = f^i_{\mu\nu} v^\mu v^\nu. \tag{2.12}$$

The second term in the deviation function does not depend on the acceleration at the physical state $\psi = (q^\mu, v^\mu)$, while the first term is non-negative and attains its absolute minimum if $D_v v^\mu = 0$, i.e. for the autoparallel trajectories.

Let us denote the Lagrange derivative as $[L]_i = -\mathrm{d}/\mathrm{d}s(\partial_{v^i} L) + \partial_{x^i} L$. We recall that elements of the tangent space are also called *virtual velocities* in analytical mechanics. The d'Alambert–Lagrange principle asserts that the conceivable motion of a system with the Lagrangian L is an actual motion if for every moment of time [3, 2]

$$(w, [L]) = 0 \tag{2.13}$$

for all virtual velocities of the constrained motion. We take $L = (v, v)/2$ and calculate $[L]_i$ for the conceivable motion $[L]_i = -\dot v^i = -\mathrm{d}/\mathrm{d}s(\varepsilon^i_\mu v^\mu)$. For virtual velocities of the constrained motion $w^i = \varepsilon^i_\mu w^\mu$, equation (2.13) assumes the form $(w, [L]) = -g_{\mu\nu} w^\nu D_v v^\mu = 0$. For an arbitrary w^μ, it leads to the autoparallel equation (1.1).

There is an equivalent formulation of the d'Alambert–Lagrange principle known as Hölder's variational principle [3, 2]. A conceivable path is called a critical point of the action functional if its variation vanishes when restricted on the subspace of virtual velocities of the constrained motion. Hölder's variational principle assumes that the actual constrained motion is the critical point of the action. For an infinitesimal variation of the trajectory we have $\delta x^i(s) = u w^i(s), u \to 0$. Then $\delta S[x]/\delta x^i = \partial_u S[x + \delta x]|_{u=0} = 0$. This is equivalent to (2.13). Restricting the virtual velocity w^i to the subset specified by constraint (2.2) we arrive at the autoparallel equation (1.1).

All the conventional variational principles for anholonomic systems are applied to *non-constrained* Lagrangians, while the constraints are implemented through the restriction of path variations to a *specific* class determined by the constraints. We shall now develop this idea further and find an equivalent variational principle that applies to the Lagrangian *restricted* on the surface of constraints. We recall that for anholonomic constraints the Hamilton variational principle leads to wrong equations of motion, when applied to the Lagrangian restricted on the constraint surface. In particular, we shall find a variational principle for the Lagrangian (1.3) or (1.4) that leads to the autoparallel equation (1.1).

Before we proceed let us make a remark. Our treatment of the autoparallel motion as an anholonomically constrained motion is somewhat unconventional. Typically, anholonomic constraints are imposed directly on positions and velocities of the unconstrained motion, i.e. on x^i and v^i (e.g. a skater on an inclined plane, a rolling ball on a rough surface, etc, [3, 13]). They can be regarded as restrictions on the *initial* values of velocities and positions so that solutions of equations of motion depend on the less number of parameters. A historical reason for such a treatment is that the positions x^i can be measured and have a natural physical meaning in many concrete anholonomic systems [13]. It is important to observe that for anholonomic constraints there exists no subspace in the original configuration space that could be identified with the physical configuration space formed by initial values of *positions* allowed by the constraints. In our anholonomic system, we have explicitly introduced a physical configuration space M, while the auxiliary Euclidean space is used only to formulate a dynamical principle that specifies the motion in M. Since anholonomic constraints do not allow us to regard M as a submanifold in the configuration space $\mathbb{R}^n$ of the unconstrained system, we have imposed constraints by restricting paths in $\mathbb{R}^n$ to a subclass permitted by the constraints, i.e. through the embedding of all trajectories of physical degrees of freedom $\mathcal{P}(M)$ into $\mathcal{P}(\mathbb{R}^n)$. This is always possible for any type of constraints.

3. Variational principle on manifolds

Here we shall give a formulation of the variational principle on manifolds that is convenient for the subsequent generalization to anholonomic systems.

Consider a vector field $w^\mu(q)$ on M satisfying the boundary condition

$$w^\mu(q_1) = w^\mu(q_2) = 0 \tag{3.1}$$

and an action functional $S = \int \mathrm{d}s\, L(v, q)$ for any trajectory connecting the points $q^\mu_{1,2}$. We *define* a variation of the action relative the vector field w^μ by

$$\delta_w S = \int \mathrm{d}s \left(\frac{\partial L}{\partial v^\mu} \mathrm{d}_w v^\mu + \frac{\partial L}{\partial q^\mu} w^\mu \right). \tag{3.2}$$

The derivative $\mathrm{d}_w v^\mu$ of the velocity along the variation vector field specifies the variation of v^μ. Given a trajectory $q^\mu = q^\mu(s)$, the velocity vector field v^μ is known only along the trajectory, while we need to know the behaviour of v^μ in the *vicinity* of the trajectory in order to calculate $\mathrm{d}_w v^\mu$ for a generic w^μ. We require

$$\mathcal{L}_w v^\mu \equiv \mathrm{d}_w v^\mu - \mathrm{d}_v w^\mu = 0 \tag{3.3}$$

where $\mathcal{L}_w v^\mu = -\mathcal{L}_v w^\mu$ denotes the Lie derivative on M. Thus, for any Lagrangian, being a function on the tangent bundle TM, the smooth vector field w^μ on M determines a variation of the position, and equation (3.3) specifies the variation of the second independent coordinate on TM, the velocity v^μ. Equation (3.3) can be given a tensor form symmetrical relative the velocity and variation vector fields

$$\mathcal{L}_w v^\mu = \mathcal{L}_v w^\mu. \tag{3.4}$$

The Euler–Lagrange equation follows from $\mathrm{d}_v w^\mu = \mathrm{d}w^\mu/\mathrm{d}s$, the boundary condition (3.1) and $\delta_w S = 0$ that should hold for any w^μ. Since the Lie derivative of a tensor is a tensor [5], equations (3.3) and (3.4) are covariant under general coordinate transformations, and so are the corresponding equations of motion if the Lagrangian is a scalar.

The above geometrical formulation of the variational principle is equivalent to the conventional one where the variation of the action is defined via smooth path variations.

Indeed, equation (3.3) implies that for any trajectory $q^\mu(s)$ with fixed endpoints there exists a one-parameter family of trajectories $q^\mu(s,u) \equiv q^\mu_u(s)$ with the same endpoints such that $q^\mu(s,u=0) = q^\mu(s)$ and $w^\mu = \partial_u q^\mu(s,u)$ in the vicinity of the trajectory $q^\mu(s)$. In other words, there exists a local coordinate net $q^\mu(s,u)$ such that v^μ and w^μ are tangent vectors for the coordinate lines $u =$ constant and $s =$ constant, respectively. The variable u plays the role of the variation parameter so that $\delta_w S[q] = \mathrm{d}/\mathrm{d}u\, S[q_u]|_{u=0}$.

If the velocity variation is specified by means of a new principle, other than a smooth deformation of the path, condition (3.3) would become modified and, therefore, the variational principle would yield new equations of motion. To obtain such a new principle for the autoparallel motion, we use the variational principles for anholonomic systems discussed in section 2. If condition (3.3) is dropped, then there exists no coordinate net $q^\mu(s,u)$ such that $w^\mu = \partial_u q^\mu(s,u)$ if $v^\mu = \partial_s q^\mu(s,u)$. For this reason the variation (3.2) will be called non-coordinate or anholonomic. Note that w^μ remains a smooth vector field on M.

4. Covariant variational principle

On a path $q^\mu(s)$ with fixed endpoints, consider a vector field w^μ that satisfies the boundary condition (3.1). Its image $w^i(s) = \varepsilon^i_\mu(q(s))w^\mu(q(s))$ determines a variation vector field on the image trajectory $x^i(s)$. By construction the variation vector field $w^i(s)$ belongs to the class of virtual velocities allowed by the constraints. The variation of the velocity vector $\mathrm{d}_w v^\mu$ is to be determined by Hölder's variational principle (2.13). We assume that the variation (3.2) of the unconstrained action (μ is to be replaced by the Cartesian index i) is equal to the Hölder variation (2.13) when the former is restricted on the constrained surface

$$\delta_w S = \int \mathrm{d}s\, (w, [L]). \tag{4.1}$$

Integrating by parts in the right-hand side of (4.1) and making use of the boundary condition (3.1) we obtain

$$\left(\mathrm{d}_w v, \frac{\partial L}{\partial v}\right) = \left(\mathrm{d}_v w, \frac{\partial L}{\partial v}\right). \tag{4.2}$$

The integration has been omitted because (4.1) holds for an arbitrary variation vector field allowed by the constraints. On the constraint surface, we have $\partial_{v^i} L = \varepsilon^\mu_i \partial_{v^\mu} L$, $\varepsilon^\mu_i = \delta_{ij} g^{\mu\nu} \varepsilon^j_\nu$ and $w^i = \varepsilon^i_\mu w^\mu$. Relation (4.2) leads to

$$(\mathrm{d}_w v - \mathrm{d}_v w, \varepsilon^\mu) = 0 \tag{4.3}$$

since $\partial_{v^\mu} L$ is also arbitrary. From the compatibility condition (2.5) it follows that

$$(\mathrm{d}_w v, \varepsilon^\mu) = D_w v^\mu \tag{4.4}$$

and, similarly,

$$(\mathrm{d}_v w, \varepsilon^\mu) = D_v w^\mu. \tag{4.5}$$

Equation (4.3) leads to the sought-for condition that specifies the variation of the velocity v^μ

$$D_w v^\mu = D_v w^\mu. \tag{4.6}$$

It can be written as

$$\mathcal{L}_w v^\mu = \mathrm{d}_w v^\mu - \mathrm{d}_v w^\mu = 2S^\mu{}_{\sigma\nu} v^\nu w^\sigma. \tag{4.7}$$

The derivative $\mathrm{d}_w v^\mu$ is proportional to the difference of the vector field v^μ at two neighbouring points q^μ and $q^\mu + u w^\mu$, $u \to 0$, i.e. $u\,\mathrm{d}_w v^\mu = (v^\mu + u\,\mathrm{d}_w v^\mu) - v^\mu$. A similar interpretation holds for $\mathrm{d}_v w^\mu$. The left-hand side of equation (4.7) contains four vectors that can be combined to form a parallelogram $[w^\mu + (v^\mu + u\,\mathrm{d}_w v^\mu)] - [v^\mu + (w^\mu + u\,\mathrm{d}_v w^\mu)]$ which is not closed as follows from (4.7). Thus, Hölder's variational principle has led us to the conclusion that the velocity variation must be chosen so that the *closure failure* of the parallelogram formed by the velocity and variation vectors would be proportional to the torsion. Note also that the closure failure of the parallelogram induced by the parallel transport of any two vector fields along one another (4.6) is also used *to define* the torsion on a manifold [5].

Let us take a Lagrangian $L = L(v, q)$ on TM and find the equation of motion resulting from the new variational principle. Substituting $\mathrm{d}_w v^\mu$ into (3.2) and using the boundary condition (3.1) to integrate by parts, we obtain

$$\delta_w S = \int \mathrm{d}s\, w^\mu \left([L]_\mu + 2S^\nu{}_{\mu\sigma} \frac{\partial L}{\partial v^\nu} v^\sigma \right). \tag{4.8}$$

The modified Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial v^\mu} - \frac{\partial L}{\partial q^\mu} - 2S^\nu{}_{\mu\sigma} \frac{\partial L}{\partial v^\nu} v^\sigma = 0. \tag{4.9}$$

To obtain equation (4.9) for the autoparallel motion in spaces with torsion, the noncommutativity of the variation and time derivative has been first postulated in [14] and later also considered in [15, 16].

Taking the Lagrangian $L = (v, v)/2$ or $L = -\sqrt{(v, v)}$ in the auxiliary Euclidean space and restricting it on the constraint surface $v^i = \varepsilon^i_\mu v^\mu$, we obtain the action (1.3) or (1.4). By construction the variational principle $\delta_w S = 0$, where the variation on TM is determined by (4.6), should yield the autoparallel equation (1.1). It is not hard to be convinced that the modified Euler–Lagrange equation (4.9) indeed leads to (1.1).

A few remarks are in order. Our derivation of the condition (4.6) does not rely on whether the unconstrained Lagrangian explicitly depends on the auxiliary Cartesian variables x^i or not. The terms involving $\partial_{x^i} L$ in equation (4.1) are cancelled. For this reason the final condition (4.6) does not depend on the form of the auxiliary unconstrained Lagrangian and may be applied to any Lagrangian on the physical configuration space.

Condition (4.6) is covariant under general coordinate transformations on M and has a transparent geometrical meaning. A *covariant* variation of v^μ along w^μ is equal to a covariant variation of w^μ along v^μ. All the modification of (3.4) we have made is that the Lie variation has been replaced by the covariant variation. It is quite remarkable that for tensors on a manifold there exists only *two* independent variations that produce tensors out of tensors and involve a displacement [5, p 336]: the Lie variation $\mathcal{L}_w$ and the covariant variation D_w. Thus, two geometrically distinguished curves on a manifold, geodesics and autoparallels, can be associated with the two independent variations available on the manifold equipped with a connection compatible with the metric.

If the torsion is not zero, the variations induced by operators D_w and D_v are non-coordinate (or anholonomic) because the basis D_μ in $T_q M$ is a non-coordinate basis [17]. Indeed, assume that there exists a coordinate net $q^\mu = q^\mu(s, u)$ such that the relation $\partial_{s,u} = D_{v,w}$ holds. Taking F to be a scalar, from (2.9) we derive

$$[D_\mu, D_\nu] F = -2S^\sigma{}_{\mu\nu} D_\sigma F. \tag{4.10}$$

The curvature term does not contribute to the commutator (4.10) because F is a scalar. Thus, $[\partial_s, \partial_u] \neq 0$, and there is no coordinate system for which the covariant derivatives

$D_{v,w}$ play the role of the translation operators along the coordinate lines.

The use of a non-coordinate basis in the tangent space to determine a variation of the action is not something unusual in mechanics. Some analogy can be made with Poincaré's variational principle in non-inertial reference frames (e.g. a rigid body in the body-fixed reference frame) [11, 3]. One looks for a *reformulation* of the Hamilton variational principle in a *non-coordinate* basis in the tangent space of M. The coordinate basis in T_qM can always be chosen as ∂_μ so that a variation of $F(q)$ is $\delta_\mu F \sim \partial_\mu F$. We can also assume another basis $e_{\mu'} = e^{\mu}_{\mu'}(q)\partial_\mu$ in the tangent space to determine a variation of any quantity on M. In general, this basis is non-commutative and, hence, non-coordinate

$$[e_{\mu'}, e_{\nu'}] = 2C^{\sigma'}{}_{\mu'\nu'}e_{\sigma'} \tag{4.11}$$

where the structure functions of the Lie algebra (4.11) are coefficients of the object of anholonomity [5]. In the new basis we have $v^\mu = e^{\mu}_{\mu'}v^{\mu'}$, $e^{\mu}_{\mu'}e^{\mu'}_{\nu} = \delta^{\mu}_{\nu}$. The components $v^{\mu'}$ of the velocity vector field in a non-coordinate basis are called quasivelocities because there is no $q^{\mu'}(q)$ such that $v^{\mu'} = \dot{q}^{\mu'}$. The problem is to find a variational principle for the Lagrangian where the velocity components are taken in the non-coordinate basis, $\tilde{L}(v^{\mu'}, q^\mu) = L(v^\mu, q^\mu)$. It is solved by rewriting equations (3.2) and (3.3) in the non-coordinate basis. Equation (3.3) assumes the form (4.6) where μ is replaced by μ' and the covariant derivative is taken relative to the connection, $\Gamma^{\sigma'}{}_{\mu'\nu'} = e^{\sigma'}_{\sigma}e^{\mu}_{\mu'}\partial_\mu e^{\sigma}_{\nu'}$, induced by going over to the new basis. Such a connection has an antisymmetric part equal to the object of anholonomity. The variational principle $\delta S/\delta w^{\mu'} = 0$ yields celebrated Poincaré's equations. They have the form (4.9) where $L \to \tilde{L}$, $\partial_{v^\mu} \to \partial_{v^{\mu'}}$, $\partial_{q^\mu} \to e^{\mu}_{\mu'}\partial_{q^\mu}$ and $S^{\sigma}{}_{\mu\nu} \to C^{\sigma'}{}_{\mu'\nu'}$.

As we see, Poincaré's variational principle is based on a *coordinate* variation (a smooth deformation of the path), but the variation of the velocity components is taken in a non-coordinate basis in the tangent space. Thus, no torsion force can be gained by considering a variational principle in an anholonomically transformed basis. By definition [5], the torsion transforms as a tensor $S^{\sigma'}{}_{\mu'\nu'} = e^{\sigma'}_{\sigma}e^{\mu}_{\mu'}e^{\nu}_{\nu'}S^{\sigma}{}_{\mu\nu} = (\Gamma^{\sigma'}{}_{\mu'\nu'} - \Gamma^{\sigma'}{}_{\nu'\mu'})/2 - C^{\sigma'}{}_{\mu'\nu'}$. If the torsion is zero in one basis it is zero in any other. The covariant variational principle always induces the torsion force because the condition (4.6) is covariant under all (coordinate and non-coordinate) transformations of the basis in the tangent space. However, it should be kept in mind that the variation specified by the condition (4.6) is no smooth deformation of the path with fixed ends if the torsion does not vanishes.

As a final remark, we shall point out that anholonomic variations in analytical mechanics have also been introduced by Sedov [12] to study dynamics of dissipative systems (they are examples of non-Lagrangian systems). He also proposed an anholonomic variational principle for such mechanical systems and considered its applications.

5. Noether's theorem

In Lagrangian mechanics first integrals of motion can be obtained from Noether's theorem. The covariant variational principle has led us to the new equations of motion (4.9). The presence of the torsion force should affect Noether's integrals of motion. Therefore it is natural to expect Noether's theorem to be modified.

Consider a one-parameter group of diffeomorphism on a manifold. Given a trajectory $q^\mu(s)$, a vector field ω^μ determines a smooth deformation of the trajectory under the one-parameter group of diffeomorphism on the manifold

$$\mathrm{d}_\omega q^\mu = \omega^\mu \qquad \mathrm{d}_\omega v^\mu = \dot{\omega}^\mu. \tag{5.1}$$

Let the Lagrangian be invariant under the transformations (5.1) up to a total time derivative

$$\mathrm{d}_\omega L = \frac{\mathrm{d}\Phi}{\mathrm{d}s} \qquad \Phi = \Phi(v, q, s). \tag{5.2}$$

If the motion is determined by the Euler–Lagrange equation, then in accordance with Noether's theorem [3], the system possesses the integral of motion

$$\frac{\mathrm{d}I}{\mathrm{d}s} = 0 \qquad I = \frac{\partial L}{\partial v^\mu}\omega^\mu - \Phi. \tag{5.3}$$

The proof follows from relation (5.2) that should be written in the form

$$\frac{\mathrm{d}I}{\mathrm{d}s} + [L]_\mu \omega^\mu = 0. \tag{5.4}$$

For the actual motion $[L]_\mu = 0$, thus leading to the conservation law (5.3). Similarly, expressing $[L]_\mu$ from (4.9) and substituting it into (5.4), we derive modified Noether's theorem

$$\frac{\mathrm{d}I}{\mathrm{d}s} = 2S^\nu{}_{\mu\sigma}\frac{\partial L}{\partial v^\nu}v^\sigma\omega^\mu. \tag{5.5}$$

In particular, for the time translation symmetry we have $\omega^\mu = v^\mu$ and $\Phi = L$. The right-hand side of (5.5) vanishes since the torsion tensor is antisymmetric. The corresponding integral of motion is the system energy. On the other hand, assuming the Lagrangian to be invariant under spatial translations and rotations (e.g. $L = \frac{1}{2}v^2$, $g_{\mu\nu} = \delta_{\mu\nu}$), we observe that the momentum and the angular momentum are no longer integrals of motion for a generic torsion tensor.

One can give two additional equivalent formulations of Noether's theorem. Assume that under the transformations (5.1) the following relation holds

$$\mathrm{d}_\omega L = \frac{\mathrm{d}\Phi}{\mathrm{d}s} - 2S^\nu{}_{\mu\sigma}\frac{\partial L}{\partial v^\nu}v^\sigma\omega^\mu. \tag{5.6}$$

Then (5.3) is an integral of motion. Clearly, to achieve (5.6), the vector field ω^μ should, in general, depend on the torsion. Although the torsion force violates the Noether conservation law as follows from (5.5), it may also admit new torsion-dependent integrals of motion†. To illustrate our statement, consider two-dimensional motion in the constant metric and torsion fields, $S^\mu{}_{\nu\sigma} = \gamma^\mu T_{\nu\sigma}$, where $T_{\nu\sigma} = -T_{\sigma\nu}$ is the generator of SO(2), $T_{12} = 1$, and $\partial_\mu\gamma^\nu = 0$, $g_{\mu\nu} = \delta_{\mu\nu}$. The Lagrangian $L = v_\mu^2/2$ exhibits the translational symmetry, but this symmetry does not lead to the conservation of the momentum components as one might see from (5.5). Nonetheless, we may solve equation (5.6) relative ω^μ and find new integrals instead of the Noether's integrals. The solution is $\Phi = 0, \omega^\mu = [\exp(\varphi T)]^\mu_\nu a^\nu$, where a^μ is an arbitrary constant vector and $\varphi = 2\delta_{\mu\nu}\gamma^\mu q^\nu$. Since a^μ is arbitrary, we have two independent integrals of motion

$$I_1 = v_1\cos\varphi - v_2\sin\varphi \qquad I_2 = v_1\sin\varphi + v_2\cos\varphi. \tag{5.7}$$

Integrals (5.7) comprise two independent first integrals for the two-dimensional autoparallel motion in the homogeneous metric and torsion fields. We also remark that $I_1^2 + I_2^2 = 2E$ where $E = L$ is the energy.

Instead of modifying the transformation law of the Lagrangian, one can modify the transformation law of its arguments, the generalized coordinates and velocities. Set

$$\mathrm{d}_\omega q^\mu \equiv \omega^\mu \qquad \mathrm{d}_\omega v^\mu \equiv \mathrm{d}_v\omega^\mu + 2S^\mu{}_{\nu\sigma}\omega^\nu v^\sigma. \tag{5.8}$$

† This conclusion seems to be contrary to the one made in [16].

If the Lagrangian is invariant under these transformations up to a total time derivative (see (5.2)), then (5.3) is an integral of motion. Attention should be paid to the fact that in contrast to the conventional formulation of Noether's theorem [3], the transformation law (5.8) determines no smooth deformation of the path $q^\mu(s)$ on a manifold and, in this sense, is not induced by any diffeomorphism on M. In fact, relation (5.8) is a postulate that specifies the transformation law of *two independent* coordinates of the tangent bundle. One first calculates $\mathrm{d}_\omega L(v,q)$, then one sets $v^\mu = \dot{q}^\mu$ and looks for such $\omega^\mu = \omega^\mu(v,q)$ that (5.2) holds for some Φ. This procedure, although being somewhat unusual, may turn out to be useful in seeking the integrals of motion for the modified Euler–Lagrange equations (4.9).

As an example, consider the autoparallel motion in the teleparallel space (zero Riemann–Cartan curvature and non-zero torsion). Such spaces are used to describe a crystal with topological defects [18]. In the anholonomic system discussed in section 2, we set $f^i_{\mu\nu} = 0$, then the curvature vanishes, while the torsion does not. We reduce either of the actions (1.3) and (1.4) on the constraint surface and obtain $L = L(v,q)$. Now it is not hard to verify that the transformations (5.8) with $\omega^\mu = (\varepsilon^\mu, a)$, a^i arbitrary constants, leaves the Lagrangian invariant ($\Phi = 0$). Since a^i are arbitrary, the quantities

$$I^i = \varepsilon^i_\mu(q)v^\mu \tag{5.9}$$

are the integrals of motion. Thus, the autoparallel motion in the teleparallel spaces can be characterized by a simple property: The velocity components taken in the special non-coordinate basis ($e_i = \varepsilon^\mu_i \partial_\mu$ in T_qM) that is transported parallel ($D_\mu \varepsilon^i_\nu = 0$) are the integrals of motion ($\dot{I}^i = 0$). This is, in general, not the case for non-teleparallel spaces.

6. A Lagrangian formalism for autoparallels

As has been pointed out in the introductionary remarks, there might exist a non-local action whose extrema (relative to the conventional variational principle) would contain autoparallels. In fact, there are infinitely many such actions. We shall give one of the possible actions. It has a few additional properties that seems to us useful for the canonical quantization of the autoparallel motion. We require that the sought action should coincide with a local action whose extrema are geodesics when the torsion is zero, and the non-locality can be removed by adding new degrees of freedom coupled to the original variables. Thus, the autoparallel motion can be modelled by a holonomic dynamics with some auxiliary degrees of freedom which admits both a Hamiltonian formalism and the canonical quantization.

We skip the details of our derivation and just give the answer. Let S_g be a local action whose extrema are geodesics, i.e. we set

$$\frac{\delta S_g}{\delta q^\mu} = g_{\mu\nu}\bar{D}_v v^\nu \ . \tag{6.1}$$

Let us introduce an integral operator $\hat{\Lambda}$ by the relation

$$\int \mathrm{d}s' \, \Lambda_{\mu\nu}(s,s') \frac{\delta}{\delta q^\sigma(s'')} D_v v^\nu(s') = g_{\mu\sigma}\delta(s-s''). \tag{6.2}$$

Hereafter the integration is extended from the initial to final moment of time. Consider the non-local action

$$S = S_g + \int \mathrm{d}s \left(D_v v^\mu - \bar{D}_v v^\mu\right) \hat{\Lambda}_{\mu\nu} D_v v^\nu = S_g + \int \mathrm{d}s \, K^\nu{}_{\mu\lambda} v^\mu v^\lambda \hat{\Lambda}_{\nu\sigma} D_v v^\sigma . \tag{6.3}$$

Using the definitions (6.1) and (6.2), one can convince oneself that from $D_v v^\mu = 0$ follows $\delta_{q^\mu} S = 0$. So the autoparallels are extrema of the action (6.3). If the torsion is zero, the non-local term in (6.3) vanishes and $S = S_g$.

The non-locality of the action (6.3) still prevents us from developing a Hamiltonian formalism. We need a local action. Fortunately, the action (6.3) can be regarded as an effective action for physical degrees of freedom in a larger dynamical system which is described by a *local* action. Let us extend the original configuration space of the system by auxiliary variables y^μ whose dynamics is determined by the equation

$$y^\mu = g^{\mu\nu}\hat{\Lambda}_{\nu\sigma} D_v v^\sigma. \tag{6.4}$$

Consider the action

$$S = S_g + \int \mathrm{d}s[K_{\mu\nu\sigma} v^\nu v^\sigma y^\mu + \lambda_\mu (g^{\mu\nu}\hat{\Omega}_{\nu\sigma} y^\sigma - D_v v^\mu)]. \tag{6.5}$$

The operator $\hat{\Omega}$ is to be chosen so that the action (6.5) turns into the action (6.3) when it is reduced on the solution of the equation of motion for the variable λ_μ.

We have

$$g_{\mu\nu}(s)\frac{\delta S}{\delta \lambda_\nu(s)} = \int \mathrm{d}s' \Omega_{\mu\sigma}(s, s') y^\sigma(s') - g_{\mu\nu} D_v v^\nu(s) = 0. \tag{6.6}$$

From equation (6.2) it follows that the kernel of the operator $\hat{\Omega}$ can be taken in the form

$$\Omega_{\mu\sigma}(s, s'') = g_{\mu\nu}(s)\frac{\delta}{\delta q^\sigma(s'')} D_v v^\nu(s). \tag{6.7}$$

With this choice equation (6.6) becomes an ordinary linear differential equation of second order for y^μ:

$$\ddot{y}^\mu + (\Gamma^\mu{}_{\nu\sigma} + \Gamma^\mu{}_{\sigma\nu}) v^\sigma \dot{y}^\nu + y^\nu \partial_\nu \Gamma^\mu{}_{\sigma\lambda} v^\sigma v^\lambda = D_v v^\mu. \tag{6.8}$$

Its solution is a sum of a general solution to the homogeneous equation $\hat{\Omega}_{\mu\nu} y^\nu = 0$ and a special solution $y^\mu = g^{\mu\nu}\hat{\Lambda}_{\nu\sigma} D_v v^\sigma$. To recover the action (6.3), we have to supplement equation (6.8) by the initial conditions such that the homogeneous equation has only a trivial solution. This is always possible thanks to the linearity of the equation. In particular, one can take $\dot{y}^\mu = y^\mu = 0$ at the initial moment of time.

The local action (6.5) *linearly* depends on the second-order derivatives $\ddot{y}^\mu$ and $\ddot{q}^\mu$. Assuming zero boundary conditions for the variable λ_μ, we may remove the second-order time derivatives by integrating by parts, thus producing the final local Lagrangian that depends only on the coordinates and velocities in the extended configuration space and involves no higher-order time derivatives. It is important to observe that the Hessian of the Lagrangian is *not* degenerate, that is, the system exhibits no constraints. The dynamics of the auxiliary degrees of freedom y^μ and λ_μ has been chosen so that when the torsion is present, the coupling between them and the original variables q^μ causes the deviation of the trajectory $q^\mu(s)$ from the geodesic, making it the autoparallel.

In the path integral formulation of quantum mechanics, the auxiliary degrees of freedom λ_μ and y^μ must be integrated out to obtain an effective path integral for the original system. This resembles the Feynman–Vernon approach to quantum dissipative systems [19], where the non-potential friction force is generated by a special coupling of the oscillator bath to the system. In our case, the non-potential (non-Lagrangian) torsion force is modelled by a special coupling to the λ- and y-degrees of freedom. Extending this analogy further, one may expect that quantum mechanics in spaces with torsion, that favours autoparallels in the classical limit, cannot, in general, be described by the wavefunction formalism,

rather only the density matrix can be constructed as for dissipative systems. The nice property which could be expected is that in the semiclassical approximation the transition amplitude generated by the effective action (6.3) is given by the geodesic action S_g taken on the autoparallel, the non-local term of (6.3) will contribute only to quantum fluctuations because $D_v v^\mu = 0$. The canonical quantization of the model will be considered elsewhere.

7. Conclusions

We have analysed the autoparallel motion from the point of view of analytical mechanics and succeeded to represent it as a very special anholonomic constrained system. Invoking the variational principles for anholonomic dynamical systems, we have established the covariant variational principle for the autoparallels. We have also analysed a modification of Noether's theorem due to the torsion force. Finally, we have found a possible local action whose extrema determined by the conventional Hamilton variational principle contain the autoparallels for some degrees of freedom. The model can be canonically quantized by means of the Dirac formalism [1].

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